Convergence Analysis of Domain Decomposition Algorithms with Full Overlapping for the Advection-Diffusion Problems.

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Abstract

The aim of this paper is to study the convergence properties of a Time Marching Algorithm solving Advection-Diffusion problems on two domains using incompatible discretizations. The basic algorithm is first presented, and theoretical or numerical results illustrate its convergence properties.

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1 Introduction

Domain decomposition methods have become an efficient strategy for solving large scale problems on parallel computers ([1], [2], [3], [4], [5], [6]). Nevertheless, they can also be used in order to couple different models [11], [18], [19] and [21]. In this paper we will examine a domain decomposition strategy which can be applied to such situations.

This approach was introduced in order to solve several difficulties that occur in fluid mechanics. In particular, our aim is to introduce several subdomains in order to locally introduce an enriched model next to a domain boundary. For this purpose, we propose to fully overlap the subdomains and to couple the solutions through natural "friction" (Neumann) forces acting on the internal boundary of the domain, these friction forces being updated inside the time marching algorithm used for the solution of the initial problem.

The theoretical study of our method will be done on an Advection-Diffusion problem, which will serve as our model problem from now on. The analysis will be made at the continuous level, independently of any discretization strategy, which means that the derived results will be mesh independent. The use of friction (Neumann) coupling boundary conditions makes the convergence analysis somewhat different of the analysis done in Kuznetsov [22] or Rannacher [23] in their study of explicit Schwarz additive methods for time evolution parabolic problems.

In the next section we will describe this model problem. In the third section we will present our algorithm for some basic cases. The fourth section will treat the one-dimensional stationary problem. We will show also that the convergence of this method can be improved by introducing a relaxation parameter [5]. The fifth section will focus on the linear convergence of the implicit version of the coupling algorithm in the general multidimensional case. In the last section we study the numerical stability of the explicit algorithm. Practical applications of the proposed algorithm to real life CFD problems can be found in [14], [19], [20], and [21].

2 The Model Problem

Consider a bounded domain, Ω of R^n such that its boundary $\partial\Omega$ is lipschitzian and Ω_{loc} a connected domain of R^n with $\Omega_{loc} \subset \Omega$ (fig. 1). The

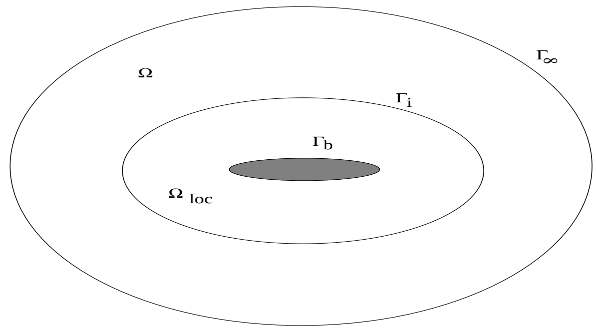

Figure 1: Description of the computational domain.

boundaries of the two subdomains are defined as follows:

$$\Gamma_b = \partial\Omega \cap \partial\Omega_{loc}, \text{ (internal boundary)}$$

$$\Gamma_i = \partial\Omega_{loc} \cap \Omega, \text{ (interface)}$$

$$\Gamma_\infty = \partial\Omega \backslash \Gamma_b. \text{ (farfield boundary)}$$

We denote by n the external unit normal vector to $\partial\Omega$ or $\partial\Omega_{loc}$.

We will make use of the following notation

$$\|v\|_{0,O} = \|v\|_{L_2(O)}$$
$$\|v\|_{s,O} = \|v\|_{H^s(O)}$$
$$|v|_{1,O} = \|\nabla v\|_{L_2(O)}$$

where O is an open bounded domain of R^n.
Let v be the velocity field inside a given incompressible flow such that:

$$\left\{ \begin{array}{rcl} \mathrm{div} v & = & 0 \text{ in } \Omega, \\ v.n & = & 0 \text{ on } \Gamma_b. \end{array} \right. \tag{1}$$

We consider the following convection-diffusion model problem:
Find φ, a real valued function, defined on Ω and satisfying

$$
\begin{cases}
\operatorname{div}(v\varphi) - \nu\Delta\varphi &= 0 \text{ in } \Omega, \\
\varphi &= \varphi^{\infty} \text{ on } \Gamma_{\infty}, \\
\varphi &= 0 \text{ on } \Gamma_b.
\end{cases}
\tag{2}
$$

Above v is the flow velocity and ν is the diffusion coefficient. Problems of this form typically occur in fluid mechanics, gas dynamics or wave propagation.

Most CFD algorithms will in fact consider the solution of this problem as the stationary solution of the evolution problem (3) described below :

Find $\phi : \Omega \times (0,T) \to R$ such that,

$$
\begin{cases}
\dfrac{\partial\phi}{\partial t} + \operatorname{div}(v\phi) - \nu\Delta\phi &= 0 \text{ in } \Omega \times (0,T), \\
\phi &= \phi^{\infty} \text{ on } \Gamma_{\infty} \times (0,T), \\
\phi &= 0 \text{ on } \Gamma_b \times (0,T), \\
\phi(0) &= \phi_0 \text{ in } \Omega.
\end{cases}
\tag{3}
$$

The general CFD algorithm consists then in integrating (3) in time until reaching a stationary solution.

3 General Algorithm

3.1 Time Continuous Case

Let us introduce the local subdomain Ω_{loc} (see fig. 1) which has as external boundary Γ_i, and let us consider the trace ϕ_{loc} of ϕ on the subdomain Ω_{loc}, as an independent variable, to which we associate an arbitrary independent initial value $\phi_{ol} \neq \phi_0|_{\Omega_{loc}}$. We now replace the evolution problem (3) by the following evolution system :

Find ϕ (resp. ϕ_{loc}) : $\Omega \to R$ (resp. $\Omega_{loc} \to R$) satisfying

$$
\begin{cases}
\dfrac{\partial\phi}{\partial t} + \operatorname{div}(v\phi) - \nu\Delta\phi &= 0 \text{ in } \Omega \times (0,T), \\
\phi &= \phi^{\infty} \text{ on } \Gamma_{\infty} \times (0,T), \\
\nu\dfrac{\partial\phi}{\partial n} &= \nu\dfrac{\partial\phi_{loc}}{\partial n} \text{ on } \Gamma_b \times (0,T),
\end{cases}
\tag{4}
$$

$$\left\{\begin{array}{rcl} \dfrac{\partial \phi_{loc}}{\partial t} + \mathrm{div}(v\phi_{loc}) - \nu\Delta\phi_{loc} & = & 0 \ \text{ in } \ \Omega_{loc} \times (0,T), \\ \phi_{loc} & = & 0 \ \text{ on } \ \Gamma_b \times (0,T), \\ \phi_{loc} & = & \phi \ \text{ on } \ \Gamma_i \times (0,T), \end{array}\right. \tag{5}$$

$$\phi(0) = \phi_0 \ \text{ in } \ \Omega, \ \ \phi_{loc}(0) = \phi_{ol} \ \text{ in } \ \Omega_{loc}. \tag{6}$$

Remark 3.1 *The global problem (4) with the initial condition (6) has no no-slip boundary condition. This suppresses the boundary layer which appears at low viscosity and facilitates the numerical solution of this problem. The boundary layers are modeled by the local problems (5)-(6) which are only to be solved on a small domain Ω_{loc}, with a very fine discretisation if needed. The two problems are only coupled by their boundary conditions and not by volumic interpolation.*

3.2 Time Discrete Case

The general algorithm that we propose for the solution of our model problem (2) is as usual to integrate in time the evolution problem (4)-(5)-(6) until we reach a stationary solution. This integration in time is then achieved by the following uncoupled semi-explicit algorithm, where the operators are treated implicitly inside each subdomain and where one of the coupling boundary conditions is treated explicitly and the other is treated implicitly:

- set $\phi^0_{loc} = \phi_{ol}$ and $\phi^0 = \phi_0$,
- then, for $n \geq 0, \phi^n_{loc}$ and ϕ^n being known, solve successively

$$\left\{\begin{array}{rcl} \dfrac{\phi^{n+1}_{loc} - \phi^n_{loc}}{\Delta t} + \mathrm{div}(v\phi^{n+1}_{loc}) - \nu\Delta\phi^{n+1}_{loc} & = & 0 \ \text{ in } \ \Omega_{loc}, \\ \phi^{n+1}_{loc} & = & \phi^n \ \text{ on } \ \Gamma_i, \\ \phi^{n+1}_{loc} & = & 0 \ \text{ on } \ \Gamma_b, \end{array}\right. \tag{7}$$

$$\left\{\begin{array}{rcl} \dfrac{\phi^{n+1} - \phi^n}{\Delta t} + \mathrm{div}(v\phi^{n+1}) - \nu\Delta\phi^{n+1} & = & 0 \ \text{ in } \ \Omega, \\ \phi^{n+1} & = & \phi^\infty \ \text{ on } \ \Gamma_\infty, \\ \nu\dfrac{\partial\phi^{n+1}}{\partial n} & = & \nu\dfrac{\partial\phi^{n+1}_{loc}}{\partial n} \ \text{ on } \ \Gamma_b. \end{array}\right. \tag{8}$$

Remark 3.2 *We have a full decoupling between (7) and (8). They can (and actually will) be discretized and solved by two independent solution techniques.*

Remark 3.3 *The fully implicit version of this method consists in replacing the condition :*

$$\phi_{loc}^{n+1} = \phi^n \ \text{ on } \ \Gamma_i$$

by the condition :

$$\phi_{loc}^{n+1} = \phi^{n+1} \ \text{ on } \ \Gamma_i.$$

The two subproblems are then coupled at each time step.

Remark 3.4 *If we replace in (8) Ω by Ω_E defined as follows :*

$$\Omega_E = \Omega \backslash \Omega_{loc},$$

and Γ_b by Γ_i, and if we set $\Delta t = \infty$, we obtain a nonoverlapping version of our strategy, which is a standard Dirichlet-Neumann algorithm [16], [17] and therefore requires a relaxation strategy to converge.

Remark 3.5 *The initial condition ϕ_{ol} is not assumed to be equal to ϕ_0 on the local subdomain Ω_{loc} because in most cases this condition is impossible to impose at the discrete level since the grid used on Ω_{loc} will be in general different from the grid used on Ω. In addition, even if we assume $\phi_{ol} = \phi_0$, we will not have $\phi_{loc}^n = \phi^n$ on Ω_{loc} unless we use the fully implicit algorithm on compatible grids.*

4 Stationary one-dimensional case

For $\Delta t = +\infty$, the above algorithm can be written :

- set $\phi_{loc}^0 = \phi_0$ and $\phi^0 = \phi_0$,
- then, for $n \geq 0, \phi_{loc}^n$ and ϕ^n being known, solve

$$\left\{ \begin{array}{rcl} \operatorname{div}(v\phi_{loc}^{n+1}) - \nu\Delta\phi_{loc}^{n+1} & = & 0 \ \text{ in } \ \Omega_{loc}, \\ \phi_{loc}^{n+1} & = & \phi^n \ \text{ on } \ \Gamma_i, \\ \phi_{loc}^{n+1} & = & 0 \ \text{ on } \ \Gamma_b, \end{array} \right. \tag{9}$$

$$\left\{\begin{array}{rcl}\operatorname{div}(v\phi^{n+1})-\nu\Delta\phi^{n+1} & = & 0 \text{ in } \Omega,\\ \phi^{n+1} & = & \phi^{\infty} \text{ on } \Gamma_{\infty},\\ \nu\dfrac{\partial\phi^{n+1}}{\partial n} & = & \nu\dfrac{\partial\phi^{n+1}_{loc}}{\partial n} \text{ on } \Gamma_b.\end{array}\right. \tag{10}$$

In one space dimension, we take the global domain Ω to be the interval $]0,1[$ of R decomposed into two fully overlapping subdomains $\Omega=]0,1[$ and $\Omega_{loc}=]h_2,1[$ with

$$0<h_2<1. \tag{11}$$

We then consider the following one dimensional problem
Find φ, a real valued function, defined on Ω and satisfying

$$\left\{\begin{array}{rcl} v\varphi'-\nu\varphi'' & = & 0 \text{ on } \Omega,\\ \varphi(0) & = & a,\\ \varphi(1) & = & b\end{array}\right. \tag{12}$$

with a constant velocity v. In this one dimensional case, the above algorithm corresponds to:

$$\left\{\begin{array}{rcl} v\varphi_2^{(n)'}-\nu\varphi_2^{(n)''} & = & 0 \quad \text{on} \quad]h_2,1[,\\ \varphi_2^{(n)}(h_2) & = & \varphi_1^{(n-1)}(h_2),\\ \varphi_2^{(n)}(1) & = & b,\end{array}\right. \tag{13}$$

$$\left\{\begin{array}{rcl} v\varphi_1^{(n)'}-\nu\varphi_1^{(n)''} & = & 0 \quad \text{on} \quad]0,1[,\\ \varphi_1^{(n)}(0) & = & a,\\ \varphi_1^{(n)'}(1) & = & \varphi_2^{(n)'}(1).\end{array}\right. \tag{14}$$

By introducing two relaxation parameters θ_1 and θ_2, we can also introduce the following variant of the above algorithm :

$$\left\{\begin{array}{rcl} v\varphi_2^{(n)'} - \nu\varphi_2^{(n)''} & = & 0 \quad \text{on} \quad]h_2,1[, \\ \varphi_2^{(n)}(1) & = & b, \\ \varphi_2^{(n)}(h_2) & = & \theta_2\varphi_1^{(n-1)}(h_2) + (1-\theta_2)\varphi_2^{(n-1)}(h_2), \end{array}\right. \tag{15}$$

$$\left\{\begin{array}{rcl} v\varphi_1^{(n)'} - \nu\varphi_1^{(n)''} & = & 0 \quad \text{on} \quad]0,1[, \\ \varphi_1^{(n)}(0) & = & a, \\ \varphi_1^{(n)'}(1) & = & \theta_1\varphi_2^{(n)'}(1) + (1-\theta_1)\varphi_1^{(n-1)'}(1). \end{array}\right. \tag{16}$$

We shall now exhibit the conditions under which the algorithm (15)-(16) converges, and those for which this convergence is optimal. For this purpose, we write the interface solution under the form

$$\varphi_1^{(n)'}(1) = \varphi'(1) + \gamma^n, \tag{17}$$

$$\varphi_2^{(n)}(h_2) = \varphi(h_2) + \delta^n, \tag{18}$$

where φ is the solution of the initial problem (12). Using the analytical solutions of the problems (15) and (16), we obtain the following induction formula

$$\begin{pmatrix} \delta^n \\ \gamma^n \end{pmatrix} = M_{IN} \begin{pmatrix} \delta^{(n-1)} \\ \gamma^{(n-1)} \end{pmatrix} . \tag{19}$$

with

$$M_{IN} = \begin{pmatrix} 1-\theta_2 & \theta_2 \dfrac{\nu}{v} e^{-(\frac{v}{\nu})}(e^{\frac{v}{\nu}h_2} - 1) \\ \dfrac{\theta_1(1-\theta_2)(\frac{v}{\nu})}{e^{(\frac{v}{\nu})(h_2-1)} - 1} & \dfrac{e^{(\frac{-v}{\nu})}\theta_1\theta_2(e^{(\frac{v}{\nu})h_2} - 1)}{(e^{(\frac{v}{\nu})(h_2-1)} - 1)} + (1-\theta_1) \end{pmatrix} \tag{20}$$

This iterative process converges if the spectral radius of the matrix M_{IN} is less than 1. A direct but tedious calculation then yields:

Lemma 4.1 *The spectral radius of the transfer matrix of the algorithm (15)-(16) is:*

$$\rho(M_{IN}) = max[\frac{1}{2}|D \pm \sqrt{D^2 - 4R}|] \tag{21}$$

with

$$D = 2 - (\theta_1 + \theta_2) + \theta_1\theta_2 e^{(-v/\nu)}(e^{(v/\nu)h_2} - 1)\frac{1}{e^{-v/\nu}e^{v(h_2/\nu)} - 1} \tag{22}$$

$$R = (1 - \theta_1)(1 - \theta_2). \tag{23}$$

From this calculation we obtain the following results:

i) When h_2 goes to 1 (nonoverlapping), D goes to $+\infty$, and then, $\rho(M_{IN})$ goes to $+\infty$. There is no-convergence at this limit.

ii) The optimal convergence is obtained in the case where all the eigenvalues of the matrix M_{IN} are zero, i.e., when : $D = 0$ and $R = 0$. The latter conditions imply in particular

$$\theta_1 = 1 \text{ or } \theta_2 = 1.$$

If we choose, in addition, $\theta_1 = \theta_2$, the condition $D = 0$ implies $h_2 = 0$. In this case the subdomain Ω_{loc} is equal to the whole domain, and the associated algorithm is no-longer of interest.

iii) The convergence of the method depends symmetrically on both relaxation parameters.

According to ii) it is reasonable to take one of the θ_i equal to 1 and call the other θ.

By setting:

$$A = 1 - \frac{e^{(-v/\nu)}(e^{(v/\nu)h_2} - 1)}{e^{(v/\nu)(h_2-1)} - 1} \tag{24}$$

we then have

$$\rho(M_{IN}) = |1 - \theta A|. \tag{25}$$

In this case, setting

$$\theta_{opt} = \{1 - \frac{e^{(-v/\nu)}(e^{(v/\nu)h_2} - 1)}{e^{(v/\nu)(h_2-1)} - 1}\}^{-1}, \tag{26}$$

which is < 1, we get the following convergence results:

Theorem 4.1 *1) The convergence is optimal (convergence in 1 iteration) if*

$$\theta = \theta_{opt}. \tag{27}$$

2) The algorithm converges for all θ in $]0, \frac{2}{A}[$.

Corollary 4.1 *1) The case without relaxation ($\theta = 1$) converges only if :*

$$\frac{2}{A} \geq 1,$$

i.e., by setting $d = 1 - h_2$ (overlapping length), only if :

$$d \geq \frac{\nu}{v} Log \frac{2}{(1 + e^{-v/\nu})} \quad \textit{(stability condition).}$$

2) When v goes to zero, we must have $d \geq \frac{1}{2}$.

Remark 4.1 *This theorem states that the application of the algorithm (15)-(16) to the time-independent problem (12) converges only if the overlapping d is sufficiently large. In the same situation, we will see that if the problem (12) can be regarded as the steady solution of a time-dependent problem and we apply our strategy to this evolution problem, the resulting algorithm will converge to the same steady solution but with less restrictions on d. This motivates the introduction of the time marching algorithm of section 3. Moreover, this time marching technique is well adapted to nonlinear problems such as those encountered in fluid mechanics (see [14], [19], [20], and [21]).*

5 Implicit Time Discretization

5.1 The General Algorithm

This section deals with the convergence analysis of the proposed algorithm in multiple dimensions when one uses the fully implicit version of our strategy (4)-(6) :

- Set $\phi^0_{loc} = \phi_{ol}$ and $\phi^0 = \phi_0$;
- then, for $n \geq 0, \phi^n_{loc}$ and ϕ^n being known, solve

$$\left\{ \begin{array}{rcl} \dfrac{\phi^{n+1} - \phi^n}{\Delta t} + \mathrm{div}(v\phi^{n+1}) - \nu\Delta\phi^{n+1} & = & 0 \quad \text{in} \quad \Omega, \\ \phi^{n+1} & = & \phi_\infty \quad \text{on} \quad \Gamma_\infty, \\ \nu\dfrac{\partial\phi^{n+1}}{\partial n} & = & \nu\dfrac{\partial\phi^{n+1}_{loc}}{\partial n} \quad \text{on} \quad \Gamma_b, \end{array} \right. \tag{28}$$

$$\left\{ \begin{array}{rcl} \dfrac{\phi^{n+1}_{loc} - \phi^n_{loc}}{\Delta t} + \mathrm{div}(v\phi^{n+1}_{loc}) - \nu\Delta\phi^{n+1}_{loc} & = & 0 \quad \text{in} \quad \Omega_{loc}, \\ \phi^{n+1}_{loc} & = & \phi^{n+1} \quad \text{on} \quad \Gamma_i, \\ \phi^{n+1}_{loc} & = & 0 \quad \text{on} \quad \Gamma_b. \end{array} \right. \tag{29}$$

5.2 Convergence Analysis

Before establishing the convergence result we shall state the preliminary results that are central to the proof of the convergence of our algorithm. The first result states the basic L^2 and H^1 local estimates.

Lemma 5.1 *We have the following estimates:*

$$\begin{aligned} \frac{1}{2\Delta t}\|\phi^{n+1} - \phi^{n+1}_{loc}\|^2_{0,\Omega_{loc}} &+ \nu|\phi^{n+1} - \phi^{n+1}_{loc}|^2_{1,\Omega_{loc}} \\ &\leq \frac{1}{2\Delta t}\|\phi^n - \phi^n_{loc}\|^2_{0,\Omega_{loc}}, \end{aligned} \tag{30}$$

$$\|\phi^{n+1} - \phi^{n+1}_{loc}\|^2_{0,\Omega_{loc}} \leq \frac{1}{1 + 2\nu\Delta tc}\|\phi^n - \phi^n_{loc}\|^2_{0,\Omega_{loc}}, \tag{31}$$

$$\|\phi^{n+1} - \phi^{n+1}_{loc}\|^2_{0,\Omega_{loc}} \leq (\frac{1}{1 + 2\nu\Delta tc})^{n+1}\|\phi^0 - \phi^0_{loc}\|^2_{0,\Omega_{loc}}, \tag{32}$$

$$\begin{aligned} \|\phi^{n+1} - \phi^{n+1}_{loc}\|^2_{0,\Omega_{loc}} &+ 2\nu\Delta t\sum_{i=p}^{n}|\phi^{i+1} - \phi^{i+1}_{loc}|^2_{1,\Omega_{loc}} \\ &\leq \|\phi^p - \phi^p_{loc}\|^2_{0,\Omega_{loc}} \; \forall p \leq n, \end{aligned} \tag{33}$$

where c is the Poincaré constant on subdomain Ω_{loc}.

Proof of lemma 5.1

Substracting (29) from (28), multiplying the result by $\phi^{n+1} - \phi_{loc}^{n+1}$ and integrating by parts over Ω_{loc}, we obtain the classical following relation:

$$\begin{aligned}\int_{\Omega_{loc}} \frac{1}{\Delta t}(\phi^{n+1} - \phi_{loc}^{n+1})^2 - \int_{\Omega_{loc}} \frac{1}{\Delta t}(\phi^{n} - \phi_{loc}^{n})(\phi^{n+1} - \phi_{loc}^{n+1}) \\ + \int_{\Omega_{loc}} \nu |\nabla(\phi^{n+1} - \phi_{loc}^{n+1})|^2 = 0.\end{aligned} \tag{34}$$

By using the Cauchy-Schwarz inequality, we obtain the estimate (30). The second estimate (31) follows by using the Poincaré inequality with c the Poincaré constant bounding the squared H^1 seminorm of any function v of $H^1(\Omega_{loc})$ with zero trace on Γ_i by its squared L^2 norm. By induction we also obtain the basic L^2 estimate (32). And finally, we obtain the estimate (33) by summing (30).

■

The above lemma states that the restriction of $\phi^{n+1} - \phi_{loc}^{n+1}$ to Ω_{loc} converges to 0 in both L^2 and H^1 norms. We shall establish now other L^2 and H^1 local estimates. Let δx^n be defined by

$$\delta x^n = \frac{(\phi^{n+1} - \phi_{loc}^{n+1}) - (\phi^{n} - \phi_{loc}^{n})}{\Delta t}, \tag{35}$$

and let G be defined by

$$G(n) = \frac{\|v\|_\infty^2}{2\nu^2}\|\phi^n - \phi_{loc}^n\|_{0,\Omega_{loc}}^2 + |\phi^n - \phi_{loc}^n|_{1,\Omega_{loc}}^2. \tag{36}$$

Lemma 5.2 *We have the following estimates:*

$$\|\delta x^n\|_{0,\Omega_{loc}}^2 \leq \frac{\nu}{\Delta t}(G(n) - G(n+1)) \tag{37}$$

$$G(n+1) \leq (\frac{1}{2\nu\Delta t} + \frac{\|v\|_\infty^2}{2\nu^2})\|\phi^{p-1} - \phi_{loc}^{p-1}\|_{0,\Omega_{loc}}^2, \ \forall p \leq n. \tag{38}$$

Proof of lemma 5.2

Substracting the two first equations in (28) and (29), multiplying the result by δx^n and integrating over Ω_{loc} we obtain

$$
\begin{aligned}
0 &= \int_{\Omega_{loc}} |\delta x^n|^2 + \int_{\Omega_{loc}} \operatorname{div}(v(\phi^{n+1} - \phi_{loc}^{n+1}))\delta x^n \\
&\quad + \nu \int_{\Omega_{loc}} \nabla(\phi^{n+1} - \phi_{loc}^{n+1}) \nabla \delta x^n \\
&\quad - \nu \int_{\partial\Omega_{loc}} \frac{\partial}{\partial n}(\phi^{n+1} - \phi_{loc}^{n+1}) \delta x^n.
\end{aligned}
$$

Using the boundary conditions in (28) and (29), and the Cauchy-Schwarz inequality we obtain

$$
\begin{aligned}
\|\delta x^n\|_{0,\Omega_{loc}}^2 &\leq \frac{1}{2}\|v\|_\infty^2 |\phi^{n+1} - \phi_{loc}^{n+1}|_{1,\Omega_{loc}}^2 + \frac{1}{2}\|\delta x^n\|_{0,\Omega_{loc}}^2 \\
&\quad + \frac{\nu}{2\Delta t}|\phi^n - \phi_{loc}^n|_{1,\Omega_{loc}}^2 - \frac{\nu}{2\Delta t}|\phi^{n+1} - \phi_{loc}^{n+1}|_{1,\Omega_{loc}}^2.
\end{aligned}
\tag{39}
$$

Using now the relation (30) (lemma 5.1) leads to the first estimate of our lemma. In fact, this estimate implies that G is a decreasing function. This property then yields

$$
\begin{aligned}
(n+2-p)G(n+1) &\leq \sum_{i=p}^{n+1} G(i) \\
&\leq \sum_{i=p}^{n+1} |\phi^i - \phi_{loc}^i|_{1,\Omega_{loc}}^2 + \frac{\|v\|_\infty^2}{2\nu^2} \sum_{i=p}^{n+1} \|\phi^i - \phi_{loc}^i\|_{0,\Omega_{loc}}^2
\end{aligned}
\tag{40}
$$

Using again the relations (30) and (31) (lemma 5.1) yields the second estimate (38). And the lemma is proved.

■

We shall establish now the global L^2 and H^1 estimates. Let $\bar{\phi}^{n+1}$ be defined as follows

$$
\bar{\phi}^{n+1} = \begin{cases} \phi^{n+1} - \varphi & \text{in} \quad \Omega \backslash \Omega_{loc}, \\ \phi_{loc}^{n+1} - \varphi & \text{in} \quad \Omega_{loc}, \end{cases}
\tag{41}
$$

with φ the solution of the stationary problem (2). By construction, $\bar{\phi}^{n+1}$ satisfies the following equations:

$$\begin{cases} \dfrac{\bar{\phi}^{n+1} - \bar{\phi}^n}{\Delta t} + \mathrm{div}(v\bar{\phi}^{n+1}) - \nu\Delta\bar{\phi}^{n+1} = 0 \quad \text{in} \quad \Omega_{loc} \cup (\Omega\backslash\Omega_{loc}), \\ \bar{\phi}^{n+1} = 0 \quad \text{on} \quad \partial\Omega, \\ \bar{\phi}^{n+1} \ \text{continuous across} \ \ \Gamma_i. \end{cases} \tag{42}$$

Let A, B_1, and B_2 be defined by the following relations

$$A = \frac{1}{1 + (\nu c - 2c_1^2)\Delta t},$$

$$B_1 = \frac{\Delta t}{c_1^2} A,$$

$$B_2 = \Delta t(\frac{\|v\|_\infty^2}{c_1^2} + \nu)A,$$

where c is the Poincaré constant and $c_1 > 0$ is an arbitrary constant as will be seen in the proof of the following lemma.

Lemma 5.3 *We have the following estimates*

$$(\frac{1}{2\Delta t} - c_1^2)\|\bar{\phi}^{n+1}\|_{0,\Omega}^2 + \frac{\nu}{2}|\bar{\phi}^{n+1}|_{1,\Omega}^2 \leq \frac{1}{2\Delta t}\|\bar{\phi}^n\|_{0,\Omega}^2 + \frac{1}{2c_1^2}\|\delta x^n\|_{0,\Omega_{loc}}^2 + (\frac{\|v\|_\infty^2}{2c_1^2} + \frac{\nu}{2})|\phi_{loc}^{n+1} - \phi^{n+1}|_{1,\Omega_{loc}}^2 \tag{43}$$

$$\|\bar{\phi}^{n+1}\|_{0,\Omega}^2 \leq A\|\bar{\phi}^n\|_{0,\Omega}^2 + B_1\|\delta x^n\|_{0,\Omega_{loc}}^2 + B_2|\phi_{loc}^{n+1} - \phi^{n+1}|_{1,\Omega_{loc}}^2 \tag{44}$$

Proof of lemma 5.3

Multiplying the equation (42) by $\bar{\phi}^{n+1}$ and integrating by parts over Ω_{loc} and $\Omega\backslash\Omega_{loc}$, and taking into account the boundary conditions in (42) we obtain the following relation:

$$\begin{aligned} &\int_\Omega \frac{\bar{\phi}^{n+1} - \bar{\phi}^n}{\Delta t}\bar{\phi}^{n+1} + \nu\int_\Omega |\nabla\bar{\phi}^{n+1}|^2 \\ &-\nu\int_{\Gamma_i} \frac{\partial}{\partial n}(\phi_{loc}^{n+1} - \phi^{n+1})\bar{\phi}^{n+1} = 0. \end{aligned} \tag{45}$$

On Ω_{loc}, $\phi_{loc}^{n+1} - \phi^{n+1}$ satisfies the following equation

$$\frac{(\phi_{loc}^{n+1} - \phi^{n+1}) - (\phi_{loc}^{n} - \phi^{n})}{\Delta t} + \mathrm{div}[v(\phi_{loc}^{n+1} - \phi^{n+1})] - \nu\Delta(\phi_{loc}^{n+1} - \phi^{n+1}) = 0. \tag{46}$$

Therefore, multiplying the above equation by $\bar{\phi}^{n+1}$, integrating by parts and using the Cauchy-Schwarz inequality we obtain

$$\begin{aligned}\nu\left|\int_{\Gamma_i} \frac{\partial}{\partial n}(\phi_{loc}^{n+1} - \phi^{n+1})\bar{\phi}^{n+1}\right| &\leq \frac{1}{2c_1^2}\|\delta x^n\|_{0,\Omega_{loc}}^2 + \frac{1}{2}c_1^2\|\bar{\phi}^{n+1}\|_{0,\Omega_{loc}}^2 \\ &+ \frac{\|v\|_\infty^2}{2c_1^2}|\phi_{loc}^{n+1} - \phi^{n+1}|_{1,\Omega_{loc}}^2 + \frac{1}{2}c_1^2\|\bar{\phi}^{n+1}\|_{0,\Omega_{loc}}^2 \\ &+ \frac{\nu}{2}|\phi_{loc}^{n+1} - \phi^{n+1}|_{1,\Omega_{loc}}^2 + \frac{\nu}{2}|\bar{\phi}^{n+1}|_{1,\Omega_{loc}}^2,\end{aligned}$$

with $c_1 > 0$ arbitrary. Combining the above inequality with (45), bounding the local norm $|f|_{i,\Omega_{loc}}$ by $|f|_{i,\Omega}$ and using the Cauchy-Schwarz inequality we obtain the estimate (43). The estimate (44) results immediately from the estimate (43) by applying the Poincaré inequality on Ω with c the Poincaré constant. And the lemma is proved.

■

Finally, we are in a position to state the main result of this section.

Theorem 5.1 *The solution of the algorithm (28)-(29) converges linearly in $H^1(\Omega)$ to the solution of the stationary problem (2), for all values of Δt and all choices of Ω_{loc}.*

Proof of theorem 5.1

Let c_1 be chosen such that $\nu c - 2c_1^2 > 0$. Using the relation (44) (lemma 5.3) we obtain by induction

$$\begin{aligned}\|\bar{\phi}^{n+1}\|_{0,\Omega}^2 \quad \leq \quad & A^p\|\bar{\phi}^{n+1-p}\|_{0,\Omega}^2 + \textstyle\sum_{i=0}^{p-1} A^i(B_1\|\delta x^{n-i}\|_{0,\Omega_{loc}}^2 + \\ & B_2|\phi_{loc}^{n+1-i} - \phi^{n+1-i}|_{1,\Omega_{loc}}^2).\end{aligned} \tag{47}$$

Since $A < 1$ by assumption on c_1, this implies

$$\begin{aligned}\|\bar{\phi}^{n+1}\|_{0,\Omega}^2 &\leq A^p\|\bar{\phi}^{n+1-p}\|_{0,\Omega}^2 + A(B_1\sum_{i=n+1-p}^{n}\|\delta x^i\|_{0,\Omega_{loc}}^2 + \\ &\quad B_2\sum_{i=n+1-p}^{n}|\phi_{loc}^{i+1}-\phi^{i+1}|_{1,\Omega_{loc}}^2).\end{aligned} \tag{48}$$

Now, using (37) (lemma 5.2) and (33) (lemma 5.1) we obtain

$$\begin{aligned}\|\bar{\phi}^{n+1}\|_{0,\Omega}^2 &\leq A^p\|\bar{\phi}^{n+1-p}\|_{0,\Omega}^2 + A(B_1\frac{\nu}{\Delta t}(G(n+1-p)-G(n+1)) + \\ &\quad B_2\frac{1}{2\nu\Delta t}\|\phi_{loc}^{n+1-p}-\phi^{n+1-p}\|_{0,\Omega_{loc}}^2).\end{aligned} \tag{49}$$

The same relation written between 0 and $n+1-p$ yields

$$\begin{aligned}\|\bar{\phi}^{n+1-p}\|_{0,\Omega}^2 &\leq A^{n+1-p}\|\bar{\phi}^o\|_{0,\Omega_{loc}}^2 + A(B_1\frac{\nu}{\Delta t}(G(0)-G(n-p+1)) \\ &\quad +B_2\frac{1}{2\nu\Delta t}\|\phi^0-\phi_{loc}^0\|_{0,\Omega_{loc}}^2).\end{aligned}$$

By combining this relation with (49), we finally obtain

$$\begin{aligned}\|\bar{\phi}^{n+1}\|_{0,\Omega}^2 &\leq A^{n+1}\|\bar{\phi}^0\|_{0,\Omega}^2 \\ &+ A^{p+1}(B_1\frac{\nu}{\Delta t}G(0) + B_2\frac{1}{2\nu\Delta t}\|\phi^0-\phi_{loc}^0\|_{0,\Omega_{loc}}^2) \\ &+ A(B_1\frac{\nu}{\Delta t}G(n+1-p) + \frac{B_2}{2\nu\Delta t}\|\phi_{loc}^{n+1-p}-\phi^{n+1-p}\|_{0,\Omega_{loc}}^2).\end{aligned}$$

Choosing p such that $n = 2p+q$, $q \geq 1$ and using (38) (lemma 5.2) we conclude that

$$\|\bar{\phi}^{n+1}\|_{0,\Omega}^2 \leq A^{n+1}C_2 + A^{p+1}C_3 + C_4\|\phi_{loc}^p - \phi^p\|_{0,\Omega_{loc}}^2, \tag{50}$$

which, from (32) (lemma 5.1), implies the linear convergence of $\|\bar{\phi}^{n+1}\|_{0,\Omega}^2$ to 0.

On the other hand by combining (37) (lemma 5.2) and (43) (lemma 5.3) we obtain

$$\begin{aligned}&(\frac{1}{2\Delta t}-c_1^2)\|\bar{\phi}^{n+1}\|_{0,\Omega}^2+\frac{\nu}{2}|\bar{\phi}^{n+1}|_{1,\Omega}^2\leq\frac{1}{2\Delta t}\|\bar{\phi}^n\|_{0,\Omega}^2\\&+\frac{\nu}{2c_1^2\Delta t}(G(n)-G(n+1))+(\frac{\|v\|_\infty^2}{2c_1^2}+\frac{\nu}{2})|\phi_{loc}^{n+1}-\phi^{n+1}|_{1,\Omega_{loc}}^2.\end{aligned}\tag{51}$$

Therefore by using (30) we obtain

$$\begin{aligned}&(\frac{1}{2\Delta t}-c_1^2)\|\bar{\phi}^{n+1}\|_{0,\Omega}^2+\frac{\nu}{2}|\bar{\phi}^{n+1}|_{1,\Omega}^2\leq\frac{1}{2\Delta t}\|\bar{\phi}^n\|_{0,\Omega}^2\\&+\frac{\nu}{2c_1^2\Delta t}(G(n)-G(n+1))+(\frac{\|v\|_\infty^2}{2c_1^2}+\frac{\nu}{2})(\frac{\|\phi_{loc}^{n}-\phi^{n}\|_{0,\Omega_{loc}}^2}{2\nu\Delta t}).\end{aligned}\tag{52}$$

Our result follows then from (38) (lemma 5.2), (32) (lemma 5.1), and the linear convergence of $\|\bar{\phi}^n\|_{0,\Omega}$.

■

5.3 Convergence of a Fixed Point Method for the Implicit Scheme

The implicit scheme proposed in this section couples the global and the local problem. To uncouple them, it is advisable to use the fixed point algorithm below :

- set $\phi_{loc,0}^0=\psi_{ol}$ and $\phi^0=\psi_0$,
- then for $k\geq 0,\phi_{k|_{\Gamma_i}}^{n+1}$ being known,

  solve

$$\left\{\begin{aligned}\frac{\phi_{loc,k+1}^{n+1}-\phi_{loc}^n}{\Delta t}+\operatorname{div}(v\phi_{loc,k+1}^{n+1})-\nu\Delta\phi_{loc,k+1}^{n+1}&=0\quad\text{in}\quad\Omega_{loc},\\ \phi_{loc,k+1}^{n+1}&=\phi_k^{n+1}\quad\text{on}\quad\Gamma_i,\\ \phi_{loc,k+1}^{n+1}&=0\quad\text{on}\quad\Gamma_b,\end{aligned}\right.\tag{53}$$

$$
\begin{cases}
\dfrac{\phi_{k+1}^{n+1} - \phi^n}{\Delta t} + \mathrm{div}(v\phi_{k+1}^{n+1}) - \nu\Delta\phi_{k+1}^{n+1} &= 0 \quad \text{in} \quad \Omega, \\
\phi_{k+1}^{n+1} &= \phi_\infty \quad \text{on} \quad \Gamma_\infty, \\
\nu\partial\phi_{k+1}^{n+1}/\partial n &= \nu\partial\phi_{loc,k+1}^{n+1}/\partial n \quad \text{on} \quad \Gamma_b.
\end{cases}
\tag{54}
$$

We will study now the algorithm (53)-(54). By setting

$$
\psi_{loc,k,q} = \phi_{loc,k+1}^{n+1} - \phi_{loc,q+1}^{n+1}, \tag{55}
$$

$$
\psi_{k,q} = (\phi_k^{n+1} - \phi_q^{n+1}), \tag{56}
$$

we observe that $\psi_{loc,k,q}$ and $\psi_{k,q}$ verify the following equations :

$$
\begin{cases}
\psi_{loc,k,q}/\Delta t + div(v\psi_{loc,k,q}) - \nu\Delta\psi_{loc,k,q} &= 0 \quad \text{in} \quad \Omega_{loc}, \\
\psi_{loc,k,q} &= \psi_{k-1,q-1} \quad \text{on} \quad \Gamma_i, \\
\psi_{loc,k,q} &= 0 \quad \text{on} \quad \Gamma_b,
\end{cases}
\tag{57}
$$

$$
\begin{cases}
\psi_{k,q}/\Delta t + div(v\psi_{k,q}) - \nu\Delta\psi_{k,q} &= 0 \quad \text{in} \quad \Omega, \\
\psi_{k,q} &= 0 \quad \text{on} \quad \Gamma_\infty, \\
\nu\dfrac{\partial\psi_{k,q}}{\partial n} &= \nu\dfrac{\partial\psi_{loc,k,q}}{\partial n} \quad \text{on} \quad \Gamma_b.
\end{cases}
\tag{58}
$$

If Δt is sufficiently small, we prove in [15] that $\psi_{k,q}$ and $\psi_{loc,k,q}$ converge linearly to zero. Hence the sequences ϕ_k^{n+1} and $\phi_{loc,k}^{n+1}$ are Cauchy sequences which converge linearly towards the unique solutions ϕ^{n+1} and ϕ_{loc}^{n+1} of the implicit scheme. This guarantees the convergence of the above fixed point algorithm.

6 Numerical Analysis of the Stability of the Algorithm (7)-(8)

In this section we focus on the application of the explicit time marching algorithm (7)-(8) studied in the previous sections to the numerical solution of the steady problem (2). We first assume that the boundary condition on Γ_b in (8) is explicit

$$(\nu \frac{\partial \phi^{n+1}}{\partial n} = \nu \frac{\partial \phi^n_{loc}}{\partial n})$$

so that the resulting algorithm is parallel (Jacobi type).

Here, Ω denotes the domain surrounding the obstacle (an ellipse in our numerical study) as described in Figure 1. The global and local domains are discretized by fully overlapping compatible finite element grids. The global mesh contains 1378 nodes and 2662 elements (see figure 7). Further the time marching algorithm is being initialized by setting ϕ_0 to zero.

In a first step, the velocity field is obtained by solving the following inviscid incompressible flow problem:

$$\begin{array}{lcl} \mathrm{div} v & = & 0, \\ \mathrm{curl}\ v & = & 0 \\ v_\infty & = & (1,0), \\ v.n & = & 0 \ \text{ on the body } \ \Gamma_b \end{array}$$

with a first order finite element method using the same global mesh.

If we set $v = 0$, the algorithm **may or may not converge** depending on the values of $\nu\Delta t$. More precisely, we observe that the algorithm converges linearly when $\nu\Delta t < \alpha_0$ and is divergent otherwise. This is graphically shown in the figures (4-7) where the values of $\frac{\|\phi^{n+1}-\phi^n\|}{\Delta t\|\phi_0\|_\infty}$ are plotted versus the iteration count n for $\nu\Delta t$ equal respectively to 10^{-6}, 10^{-1}, 1, and 10. Further, when the velocity is taken sufficiently large, the algorithm becomes unconditionally stable. In particular, the initialization of our algorithm by $\phi_0 = 0$ with $\|v_\infty\| = 1$, $\nu = 0.1$ and $\Delta t = 100$ leads to a converging algorithm (fig. 8).

By intuition such a behavior seems natural. An overestimation of the solution ϕ^n at the interface Γ_i implies an overestimation of the friction forces on Γ_b. For sufficiently small time steps, this overestimation will not affect the value of ϕ^{n+1} on Γ_i and can therefore be ignored at the next time step. If the Reynolds is sufficiently large, this error will only affect the wake region but will not have any influence at the interface Γ_i. To the contrary, for large Δt and ν, this error does affect the value of ϕ^{n+1} on Γ_i. The influence of the error on ϕ^{n+1} may be amplified throughout the iteration process.

Another variant of the algorithm consists of replacing the explicit Dirichlet condition

$$\phi_{loc}^{n+1} = \phi^n \text{ on } \Gamma_i \text{ in the algorithm (7)-(8)}$$

by the following semi-implicit condition

$$\phi_{loc}^{n+1} = \phi^{n+1} \text{ on } \Gamma_i.$$

In fact, this implies replacing the previously parallel algorithm (Jacobi like) by the sequential algorithm (Gauss-Seidel like).

When we solve the pure diffusion problem (i.e. with flow velocity $v = 0$) with $\nu = 1$ and $\Delta t = 1$ (respectively $\Delta t = 2$) we obtain a better convergence history :

- the speed of the new algorithm is linear and clearly faster than the parallel algorithm.
- the domain of convergence is moderately larger (see table 1).

To study experimentally in more details the convergence behavior of both algorithms we assume that we have a linear behavior of our algorithm, and hence that the error at the iteration n will satisfy the following inequality

$$\|\phi^{n+1} - \phi^n\|_\infty \approx K^n \|\phi^1 - \phi^0\|_\infty.$$

The algorithm converges if $K < 1$. An estimate for K can be found by considering as in table 1 the ratio

$$-\frac{1}{n}\log\frac{\|\phi^{n+1} - \phi^n\|_\infty}{\|\phi^1 - \phi^0\|_\infty} = -\log K$$

which is displayed as a function of $(\nu\Delta t)$ for $n = 14$ and different values of $V = \frac{v}{\nu}$. A negative value of this ratio means divergence of the algorithm. As expected, this ratio is positive for sufficiently small values of Δt and converges to zero as Δt goes to zero.

In this table, we observe that for $V = 0, \nu\Delta t < \alpha_0 \approx 2$, the algorithm converges. However the convergence is slow since the minimal contraction constant K_{min} (for the optimal value of $\nu\Delta t$) is close to one (see table 2). For $V = 10$, the algorithm converges for a much larger range of values of $\nu\Delta t$ and the optimal contraction constant is much smaller. This is summarized on table 2 where we have displayed the best possible contraction constants for each of the coupling algorithms and for different values of the Reynolds $V = \frac{v}{\nu}$.

$\nu\Delta t$	1/1000	1/10	1/2	2	5	10	50	1000
Gauss-Seidel $V = 0$	0.06	0.1	0.22	0.5	-0.27	-0.5	-0.75	-0.8
Jacobi $V = 0$	-	0.1	0.22	0	-0.09	-0.25	-0.4	-0.41
Gauss-Seidel $V = 10$	0.03	0.25	1.46	2.12	2.8	2.6	2.4	2.4
Jacobi $V = 10$	0.03	0.28	1.15	1.15	1.15	1.15	1.14	1.14
Jacobi $V = 1000$	0.23	2.79	2.8	2.7	2.75	2.8	-	-

Table 1: Contraction constant (in fact minus its logarithm) in function of $\nu\Delta t$ for the explicit (Jacobi) and semi-explicit (Gauss-Seidel) version of our coupling algorithm. We observe divergence for $V = 0$ and $\nu\Delta t > 2$ and convergence otherwise.

Jacobi (parallel)		Gauss-Seidel (sequential)	
V	K_{min}	V	K_{min}
0	0.85	0	0.68
10	0.50	10	0.11
10^3	0.14		

Table 2: Minimal contraction constant versus the Reynolds V for both sequential and parallel versions of the algorithm.

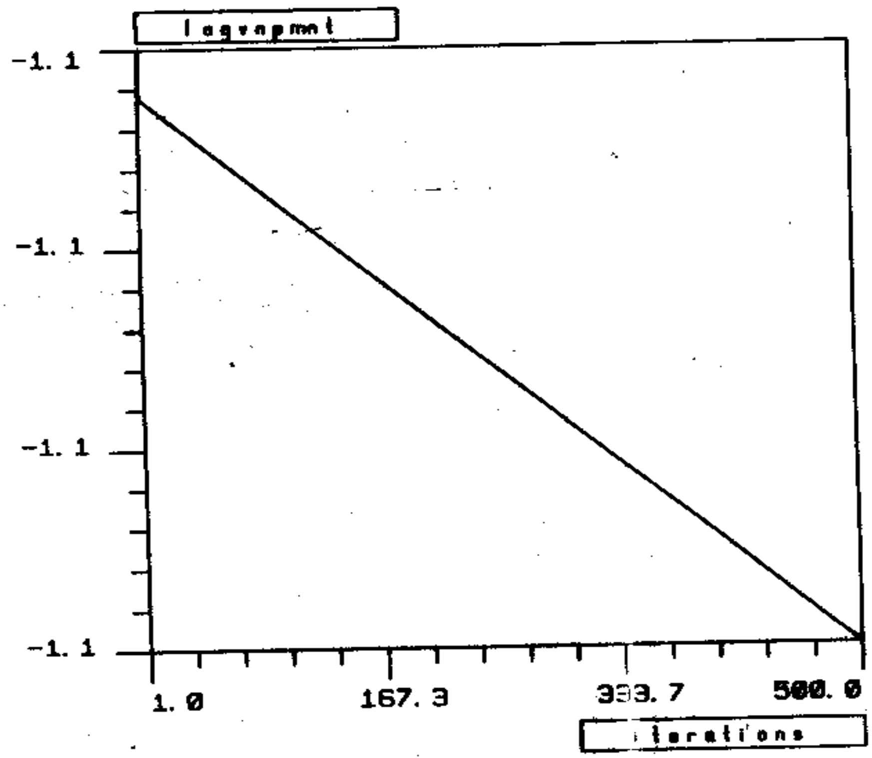

Figure 2: Convergence of the Time Marching Algorithm: $\frac{\|\phi^{n+1}-\phi^{n}\|}{\Delta t\|\phi_0\|_\infty}$ are plotted versus the iteration count n for $\nu\Delta t = 10^{-6}$, $v = 0$ (Jacobi). Observe the very slow convergence.

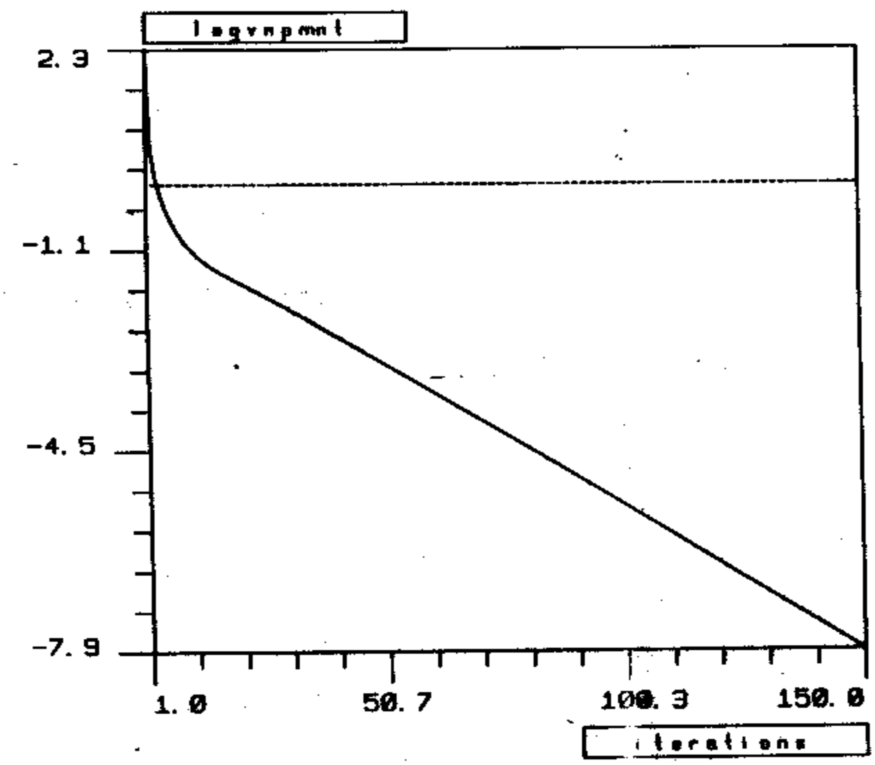

Figure 3: Convergence of the Time Marching Algorithm: $\frac{\|\phi^{n+1}-\phi^{n}\|}{\Delta t\|\phi_0\|_\infty}$ are plotted versus the iteration count n for $\nu\Delta t = 10^{-1}$, $v = 0$ (Jacobi).

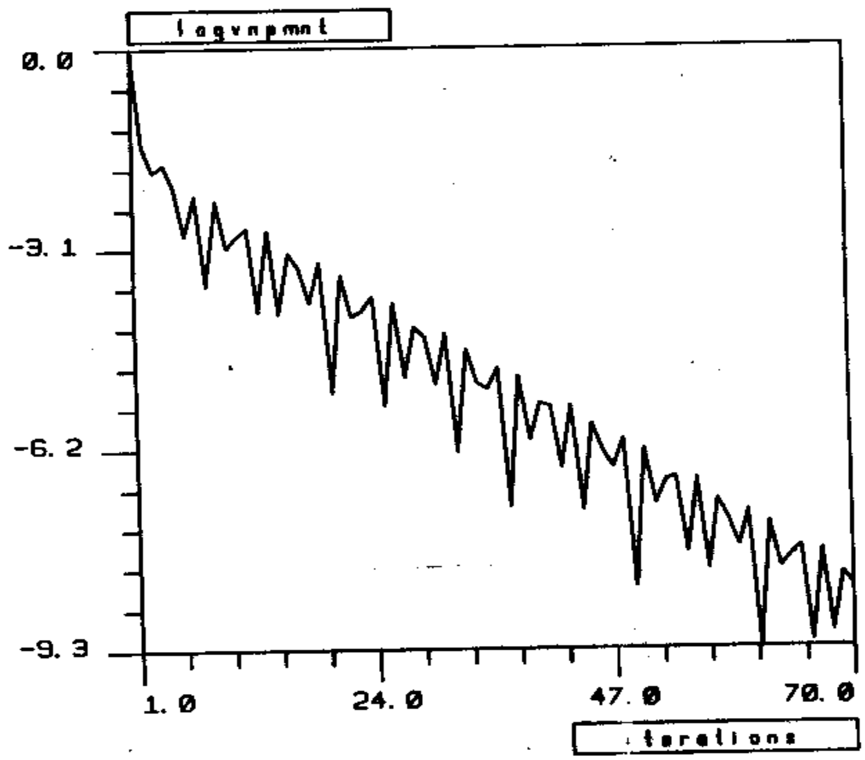

Figure 4: Convergence of the Time Marching Algorithm: $\frac{\|\phi^{n+1}-\phi^n\|}{\Delta t\|\phi_0\|_\infty}$ are plotted versus the iteration count n for $\nu\Delta t = 1$, $v = 0$ (Jacobi).

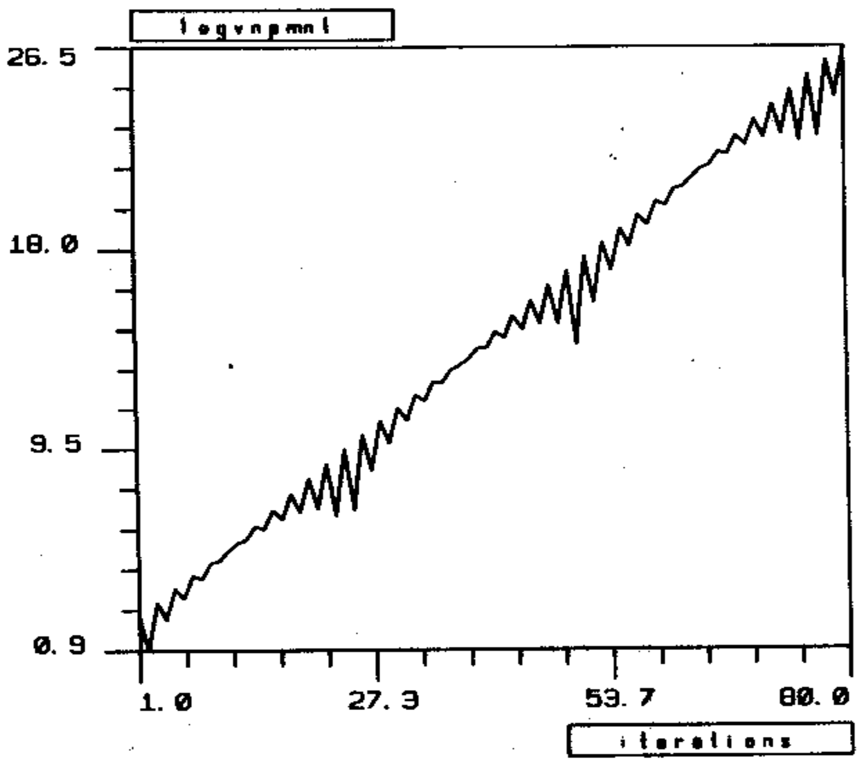

Figure 5: Divergence of the Time Marching Algorithm: $\frac{\|\phi^{n+1}-\phi^n\|}{\Delta t \|\phi_0\|_\infty}$ are plotted versus the iteration count n for $\nu\Delta t = 10$, $v = 0$ (Jacobi).

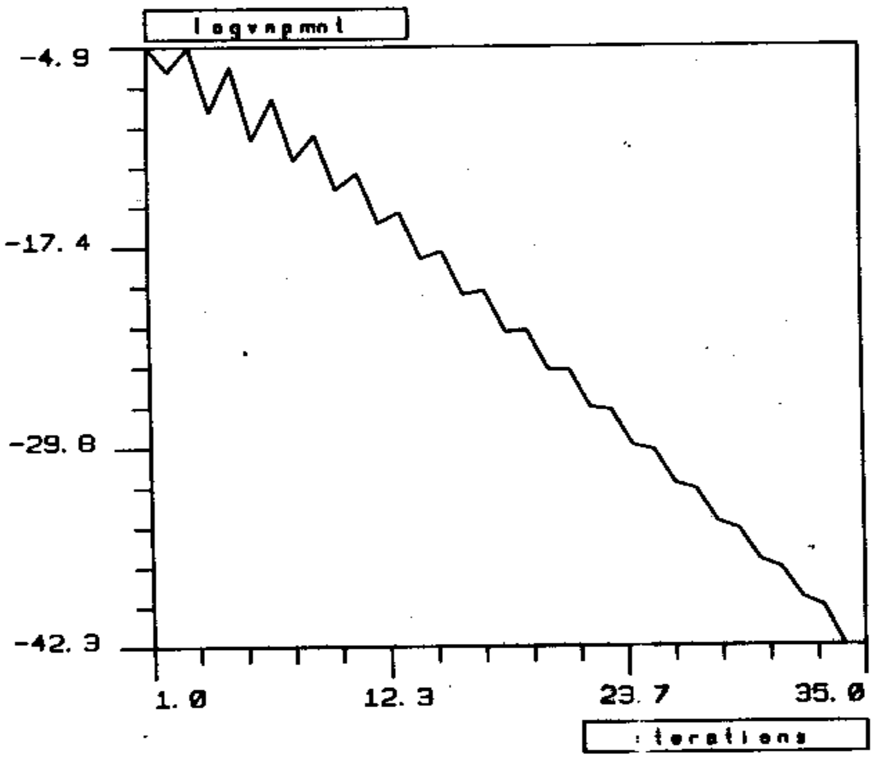

Figure 6: Convergence of the Time Marching Algorithm: $\frac{\|\phi^{n+1}-\phi^n\|}{\Delta t\|\phi_0\|_\infty}$ are plotted versus the iteration count n for $\nu\Delta t = 10$ and the flow velocity is equal to 1 (Jacobi).

7 Conclusion

We have analysed the convergence properties of a standard time marching algorithm for solving a domain decomposed advection-diffusion problem with full overlapping and coupling by friction. We were able to prove theoretically the unconditional stability and linear convergence of the fully implicit algorithm (§5).

When using the uncoupled semi-explicit algorithm in the general case, numerical evidence indicate that this algorithm is unstable for large values of Δt and small overlapping, and that it becomes linearly convergent when Δt is below a Reynolds dependent threshold (§7). This conditional stability is not a real issue for practical CFD problems because most solvers already require to use small time steps inside each domain. Nevertheless, it would be nicer to derive an uncoupled unconditionally stable version of the present time marching algorithm.

References

[1] R. Glowinski, G. Golub and J. Periaux (eds), *Proceedings of the First International Symposium on Domain Decomposition Methods for Partial Differential Equations, Paris, France, January 7-9, 1987*, (SIAM, Philadelphia, 1988).

[2] T. Chan, R. Glowinski, J. Periaux and O. Widlund (eds), *Proceedings of the Second International Symposium on Domain Decomposition Methods for Partial Differential Equations, Los Angeles, California, January 1988*, (SIAM, Philadelphia, 1989).

[3] T. Chan and R. Glowinski (eds), *Proceedings of the Third International Symposium on Domain Decomposition Methods for Partial Differential Equations, Houston, Texas, March 20-22, 1989*, (SIAM, Philadelphia, 1990).

[4] R. Glowinski, Y. Kuznetsov, G. Meurant, J. Periaux and O. Widlund (eds), *Proceedings of the Fourth International Symposium on Domain Decomposition Methods for Partial Differential Equations, Moscow, June 1990*, (SIAM, Philadelphia, 1991).

[5] T. Chan, D. Keyes, G. Meurant, S. Scroggs and R. Voigt (eds), *Proceedings of the Fifth International Symposium on Domain Decomposition Methods for Partial Differential Equations, Norfolk, May 1991*, (SIAM, Philadelphia, 1992).

[6] A. Quarteroni (ed), *Proceedings of the Sixth International Symposium on Domain Decomposition Methods for Partial Differential Equations, Como, June 1992*, (AMS, Providence, 1994).

[7] Y. Achdou and O. Pironneau, A fast solver for Navier-Stokes equations in the laminar regime using mortar finite element and boundary element methods, *Technical Report 93-277* (Centre de Mathématiques Appliquées, Ecole Polytechnique, Paris, 1993).

[8] A. D. Aleksandrov, Majoration of solutions of second order linear equations, *Vestnik Leningrad Univ. 21, 5-25(1966) English translation in Amer. Math. Soc. Transl. (2) 68, 120-143(1968).*

[9] I. Ya. Bakel'man, Theory of quasilinear elliptic equations. *Siberian Math. J. 2, 179-186(1961).*

[10] J. Bramble, R. Ewing, R. Parashkevov and J. Pasciak, Domain decomposition methods for problems with partial refinement *SIAM J. Sci. Stat. Comp.* **13** 397–410 (1992).

[11] C. Canuto and A. Russo, On the Elliptic-Hyperbolic Coupling. I: the Advection Diffusion Equation via the ξ-formulation, *Math. Models and Meth. Appl. Sciences* (to appear) (1993).

[12] D. Gilbarg and N. S. Trudinger, Elliptic partial differential equations of second order. *Berlin-Heidelberg-New York, Springer Verlag 1977.*

[13] W. Gropp and D. Keyes, Domain Decomposition Methods in Computational Fluid Dynamics. *Int. J. Num. Meth. Fluids* **14** 147-165 (1992).

[14] P. Le Tallec and M. D. Tidriri, Coupling Navier-Stokes and Boltzmann. Submitted to J. Comp. Phy.

[15] P. Le Tallec and M. D. Tidriri, Analysis of the explicit time marching algorithm. ICASE Report No. 96-45.

[16] L. Marini and A. Quarteroni, An iterative procedure for domain decomposition methods: a finite element approach. In [1].

[17] L. D. Marini and A. Quarteroni, A relaxation procedure for domain decomposition methods using Finite Elements, *Numer. Math.* **55**, (1989) 575–598.

[18] A. Quarteroni, G. Sacchi Landriani and A Valli, Coupling of Viscous and Inviscid Stokes Equations via a Domain Decomposition Method for Finite Elements, *Technical report UTM89-287* (Dipartimento di Mathematica, Universita degli Studi di Trento, 1989).

[19] M. D. Tidriri, Couplage d'approximations et de modèles de types différents dans le calcul d'écoulements externes,*PhD thesis, University of Paris IX, 1992.*

[20] M. D. Tidriri, Domain Decomposition for Incompatible Nonlinear Models. *INRIA Research Report RR-2378, October 1994.*

[21] M. D. Tidriri, Domain decomposition for compressible Navier-Stokes equations with different discretizations and formulations. *J. Comp. Phy.* **119**, 271-282 (1995).

[22] Y. A. Kuznetsov, Overlapping Domain Decomposition Methods for Parabolic Problems. In [6].

[23] H. Blum, S. Lisky and R. Rannacher, A domain decomposition algorithm for parabolic problems, Preprint 02-08, Interdisziplinaeres Zentrum fuer Wissenschaftliches Rechen, Universitaet Heidelberg, 1992.

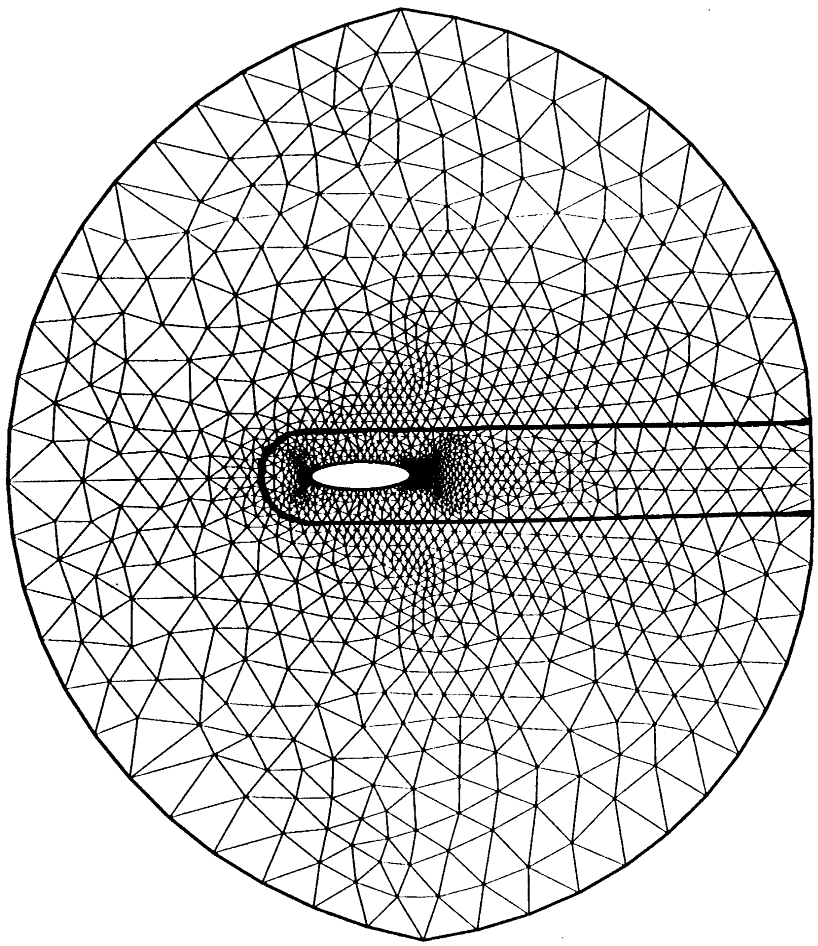

Figure 7: Description of the finite element mesh and of the local subdomain.